AS-2219

M.A./M.Sc. (First Semester) Examination, 2013

MATHEMATICS

Paper-Third : TOPOLOGY-I

1. a) A normal T_1-space is said to be T_4-space

Ex: The space (R, U) is a T_4-space

normal: A topological space is said to be normal iff for every pair L, M of disjoint $\mathfrak{J}$-closed sub-sets of X there exists $\mathfrak{J}$-open sets G and H such that $L \subset G$, $M \subset H$ and $G \cap H = \phi$.

(any valid example is also considered)

b) First countable: A first countable space is a topological space in which there exists a countable local base at each of its points

second countable: A topological space $(X, \mathfrak{J})$ is said to be second countable iff there exists a countable base for $\mathfrak{J}$.

example: (for first countable. if $x \in R$, then the collection $\{(x-\frac{1}{n}, x+\frac{1}{n}) : n \in N\}$ forms a countable base at x

(for second countable

(c) Let $X = \{a, b, c, d, e\}$ and

$\mathfrak{J} = \{X, \phi, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, e\}\}$

The boundary of $A = \{a\}$ is

Finding ext of A is ~~[illegible]~~

Finding Int of A is ϕ

boundary of $b(A)$ is

(d) Let p be a limit point of the set A iff every open neighbourhood of p contains a point of A other than 'p'.

i.e $p \in G$ implies $(G \setminus \{p\}) \cap A \neq \emptyset$ when $G \subset A$.

Now, we show that p is a limit point of $A \setminus \{p\}$

Let $G \subset A$. $p \in G$ implies that $(G \setminus \{p\}) \cap (A - \{p\}) = (G \cap A) - \{p\}$
$= G \setminus \{p\}$
$= \emptyset$

Hence 'p' is a limit point of $A \setminus \{p\}$ also.

(e)

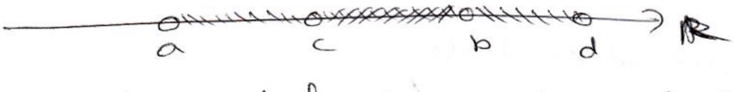
Every subspace of an indiscrete space is indiscrete

It is valid ~~for~~ showing the proof with example and explain

(f) Let $\mathcal{B}$ be the class of open-closed intervals in the real line $\mathbb{R}$.

$\mathcal{B} = \{ [a, b] : a, b \in \mathbb{Q}, a < b$

Clearly, $\mathbb{R}$ is the union of members of $\mathcal{B}$ since every rational number belongs to some open-closed intervals.



and also explain and showing not base for topology on the real line $\mathbb{R}$

(g) Smallest topology $\mathcal{T} = \{X, \emptyset\}$

largest topology $\mathcal{T}' = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$

h) Since U is open in Y, $U = Y \cap V$ for some set V open in X.

Since Y and V are both open in X, so is $Y \cap V$.

(i) Let X be a normal space; let A be a closed subset of X

(a) Any continuous map of A into the closed interval $[a, b]$ of $\mathbb{R}$ may be extended to a continuous map of all of X into $[a, b]$.

(b) Any continuous map of A into the reals $\mathbb{R}$ may be extended to a continuous map of all of X into $\mathbb{R}$.

(j) Let $X = \{a, b, c, d, e\}$

$\mathcal{J} = \{X, \phi, \{b, c, d, e\}, \{a, c, d\}, \{c, d\}, \{a\}\}$

$A = \{a, d, e\}$

$\mathcal{J}_A = \{A \cap G : G \in \mathcal{J}\}$ so the members of $\mathcal{J}_A$ are

$A \cap X = A$: $A \cap \phi = \phi$, $A \cap \{b, c, d, e\} = \{d, e\}$

$A \cap \{a, c, d\} = \{a, d\}$, $A \cap \{c, d\} = \{d\}$, $A \cap \{a\} = \{a\}$

$\therefore \mathcal{J}_A = \{A, \phi, \{a\}, \{d\}, \{a, d\}, \{d, e\}\}$

(P.T.O)

2. a) Assume that $A = C \cap Y$, where C is closed in X. Then $X - C$ is open in X, so that $(X-C) \cap Y$ is open in Y. by definition of the sub-space topology. But $(X-C) \cap Y = Y - A$. Hence $Y - A$ is open in Y, so that A is closed in Y.

Conversely, assume that A is closed in Y. Then $Y - A$ is open in Y, so that by definition it equals the intersection of an open set U of X with Y. The set $X - U$ is closed in X, and $A = Y \cap (X - U)$, so that A equals the intersection of a closed set of X with Y.

b) 1. ϕ and X are closed, because they are the complements of the open sets X and ϕ, respectively

2. Given a collection of closed sets $\{A_\alpha\}_{\alpha \in J}$ we apply DeMorgan's law $X - \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha)$.

Since the sets $X - A_\alpha$ are open by definition, the right side of this equation represents an arbitrary union of open sets, and is thus open. Therefore $\bigcap A_\alpha$ is closed

3. Similarly, if A_i is closed for $i = 1, \ldots, n$, consider the equation $X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i)$.

The set on the right side of this equation is a finite intersection of open sets and is therefore open.

Hence $\bigcup A_i$ is closed

3 a) The intersection $\mathfrak{J}_1 \cap \mathfrak{J}_2$ of any two topologies $\mathfrak{J}_1$ and $\mathfrak{J}_2$ on X is also a topology on X.

T_1: Let $X \in \mathfrak{J}_1,\ X \in \mathfrak{J}_2 \Rightarrow X \in \mathfrak{J}_1 \cap \mathfrak{J}_2$

$\phi \in \mathfrak{J}_1,\ \phi \in \mathfrak{J}_2 \Rightarrow \cancel{X}\phi \in \mathfrak{J}_1 \cap \mathfrak{J}_2$

T_2: Let $G_1 \in \mathfrak{J}_1 \cap \mathfrak{J}_2 \Rightarrow G_1 \in \mathfrak{J}_1$ and $G_1 \in \mathfrak{J}_2$

$G_2 \in \mathfrak{J}_1 \cap \mathfrak{J}_2 \Rightarrow G_2 \in \mathfrak{J}_1$ and $G_2 \in \mathfrak{J}_2$

$\therefore G_1 \in \mathfrak{J}_1,\ G_2 \in \mathfrak{J}_1 \Rightarrow G_1 \cap G_2 \in \mathfrak{J}_2$ (since $\mathfrak{J}_1$ is topology)

$G_1 \in \mathfrak{J}_2,\ G_2 \in \mathfrak{J}_2 \Rightarrow G_1 \cap G_2 \in \mathfrak{J}_1$ (since $\mathfrak{J}_2$ is topology)

$\therefore G_1 \cap G_2 \in \mathfrak{J}_2$ and $G_1 \cap G_2 \in \mathfrak{J}_1 \Rightarrow G_1 \cap G_2 \in \mathfrak{J}_1 \cap \mathfrak{J}_2$

T_3: Let $G_1 \in \mathfrak{J}_1 \cap \mathfrak{J}_2$ and $G_2 \in \mathfrak{J}_1 \cap \mathfrak{J}_2$

$\therefore G_1 \in \mathfrak{J}_1 \cap \mathfrak{J}_2 \Rightarrow G_1 \in \mathfrak{J}_1$ and $G_2 \in \mathfrak{J}_2$

$G_2 \in \mathfrak{J}_1 \cap \mathfrak{J}_2 \Rightarrow G_2 \in \mathfrak{J}_1$ and $G_2 \in \mathfrak{J}_2$

$\therefore G_1 \in \mathfrak{J}_1,\ G_2 \in \mathfrak{J}_1 \Rightarrow G_1 \cup G_2 \in \mathfrak{J}_1$ (since $\mathfrak{J}_1$ is topology)

and $G_2 \in \mathfrak{J}_2,\ G_2 \in \mathfrak{J}_2 \Rightarrow G_1 \cup G_2 \in \mathfrak{J}_2$ (since $\mathfrak{J}_2$ is topology)

Hence $G_1 \cup G_2 \in \mathfrak{J}_1,\ G_1 \cup G_2 \in \mathfrak{J}_2 \Rightarrow G_1 \cup G_2 \in \mathfrak{J}_1 \cap \mathfrak{J}_2$

$\therefore \mathfrak{J}_1 \cap \mathfrak{J}_2$ is a topology on X.

(b) T_1: Since $X' = \phi$ (which is finite) we have $X \in \mathfrak{J}$.
Also $\phi \in \mathfrak{J}$ by definition

T_2 $G_1, G_2 \in \mathfrak{J} \Rightarrow G_1', G_2'$ are finite
$\Rightarrow G_1' \cup G_2'$ is finite
$\Rightarrow (G_1 \cap G_2)'$ is finite
$\Rightarrow G_1 \cap G_2 \in \mathfrak{J}$

T_3 $G_\lambda \in \mathfrak{J}\ \forall\ \lambda \in \Lambda \Rightarrow G_\lambda'$ is finite $\forall\ \lambda \in \Lambda$
$\Rightarrow \cap \{ G_\lambda' : \lambda \in \Lambda \}$ is finite
$\Rightarrow [\cup \{ G_\lambda : \lambda \in \Lambda \}]'$ is finite
$\Rightarrow \cup \{ G_\lambda : \lambda \in \Lambda \} \in \mathfrak{J}$.

4 (a) Suppose that f is continuous. Given x and ε,

Consider the set $f^{-1}(B(f(x), \varepsilon))$

which is open in X and contains the point x. It contains some δ-ball $B(x, \delta)$ centered at x. If y is in this δ-ball, then $f(y)$ is in the ε-ball centered at $f(x)$, as desired.

Conversely, suppose that the ε-δ condition is satisfied. Let V be open in Y; we show that $f^{-1}(V)$ is open in X. Let x be a point of the set $f^{-1}(V)$. Since $f(x) \in V$, there is an ε-ball $B(f(x), \varepsilon)$ centered at $f(x)$ and contained in V. By the ε-δ condition, there is a δ-ball $B(x, \delta)$ centered at x such that $f(B(x,\delta)) \subset B(f(x), \varepsilon)$. Then $B(x, \delta)$ is a neighbourhood of x contained in $f^{-1}(V)$, so that $f^{-1}(V)$ is open.

(b) Since $\bar{d}(x,y)$ is metric (showing metric space is not needed)

To show that d and $\bar{d}$ are equivalent; it suffices to show that every d-open sphere centered at $x \in X$ contains a $\bar{d}$-open sphere centered at x and conversely.

Since

$$\bar{d}(x,y) \leq d(x,y) \quad \forall\, x, y \in X$$

It follows that

$$S(x,r) = \{y \in X : d(x,y) < r\} \subset \{y \in X : \bar{d}(x,y) < r\}$$
$$= \bar{S}(x, r)$$

for every $x \in X$ and every $r > 0$.

on the other hand, if r is positive number, we let

$$\rho = \min\{1, d(x,y)\}.$$

Then

$$\bar{S}(x,\rho) = \{y \in X : d(x,y) < \rho\} \subset \{y \in X : \bar{d}(x,y) < r\}$$

$$= S(x,r) \text{ for every } x \in X.$$

It follows that every d-open set $\bar{d}$-open and conversely,

Hence d and $\bar{d}$ are equivalent.

5. (a) $A \cap B \subset A \Rightarrow \overline{(A \cap B)} \subset \bar{A}$

and $A \cap B \subset B \Rightarrow \overline{A \cap B} \cap \bar{B}$

Hence $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$

But $\overline{A \cap B} \supset \bar{A} \cap \bar{B}$ doesn't happens.

any suitable example is considered

(b) In usual metric consider the set $A = (0,1)$ and $B = (1,2)$

Then $A \cap B = \phi$. $\bar{A} = [0,1]$ and $\bar{B} = [1,2]$

$$\bar{A} \cap \bar{B} = [0,1] \cap [1,2] = \{1\}$$

$$\overline{A \cap B} = \phi$$

$$\therefore \bar{A} \cap \bar{B} \neq \overline{A \cap B}$$

(b) $A - B \overset{c}{\supset} A \Rightarrow \overline{A-B} \overset{c}{\supset} \bar{A}$

and $A - B \overset{c}{\supset} B \Rightarrow \overline{A-B} \overset{c}{\supset} \bar{B}$

Hence $\bar{A} - \bar{B} \subset \overline{A-B}$

any suitable example for showing $\bar{A} - \bar{B} \supset \overline{A-B}$ not satisfying

is considerable

6. Regular space: A topological space $(X, \mathfrak{J})$ is said to be regular if and only if for every $\mathfrak{J}$-closed set F and every point $p \notin F$, there exist $\mathfrak{J}$-open set G and H such that

$$F \subset G, \; p \in H \text{ and } G \cap H = \emptyset$$

example Let $X = \{(x, y) \in R \times R : y \geq 0\}$
to define a topology $\mathfrak{J}$ on R as follows
For each $(p, q) \in X \; (q > 0)$, let

$$N_\varepsilon(p, q) = \{(x, y) : \sqrt{(x-p)^2 + (y-q)^2} < \varepsilon\}$$

and $$N_\varepsilon(p, 0) = \{(x, y) : \sqrt{(x-p)^2 + (y-q)^2} < \varepsilon\} \cup \{(p, 0)\}$$

For each $(p, q) \in X$, define

$$N(p, q) = \begin{cases} N_\varepsilon(p, q) : \varepsilon < 0 & \text{when } q > 0 \\ N_\varepsilon(p, 0) : \varepsilon > 0 & \text{when } q > 0 \end{cases}$$

It is easy to see that the collection $N(p, q)$ satisfies property hence there exists a topology $\mathfrak{J}$ for X such that $N(p, q)$ is a collection of basic $\mathfrak{J}$-nbds of (p, q). Hence T_3-space
(Any other suitable example is also considered)
Now last part of the problem
Showing that regular space need not be T_3-space.
Let $X = \{a, b, c\}$ and let $\mathfrak{J} = \{\emptyset, \{a\}, \{b, c\}, X\}$.
The closed subset are X, $\{b, c\}$, $\{a\}$, $\emptyset$.
Consider the closed set $\{b, c\}$ and the point a not belonging to it. Then $\{b, c\}$ and $\{a\}$ are open sets such that

$$\{b, c\} \subset \{b, c\}, \; a \in \{a\} \text{ and } \{b, c\} \cap \{a\} = \emptyset.$$

Similarly, consider the closed set $\{a\}$ and the point b not belonging to it. Then $\{a\}$ and $\{b, c\}$ are open sets such that

$$\{a\} \subset \{a\}, \; b \in \{b, c\}, \text{ and } \{a\} \cap \{b, c\} = \emptyset$$

Again for the closed set $\{a\}$ and the point c, there exist open sets $\{a\}$ and $\{b, c\}$ such that

$$\{a\} \subset \{a\},\ c \in \{b, c\} \text{ and } \{a\} \cap \{b, c\} = \phi.$$

It follows that $(X, \mathfrak{J})$ is a regular space

Since there does not exist a $\mathfrak{J}$-open set containing the point b and not containing the point c, the space is not T_3.

(b) Let $(X, \mathfrak{J})$ be a normal space and let $(Y, \mathcal{V})$ be a homeomorphic image of $(X, \mathfrak{J})$ under a homeomorphism f.

To show that $(Y, \mathcal{V})$ is also a normal space.

Let L, M be a pair of disjoint $\mathcal{V}$-closed subsets of Y.

Since f is a continuous map. $f^{-1}(L)$ and $f^{-1}(M)$ are $\mathfrak{J}$-closed subset of X.

$$f^{-1}(L) \cap f^{-1}(M) = f^{-1}(L \cap M) = f^{-1}(\phi) = \phi.$$

Thus $f^{-1}(L), f^{-1}(M)$ is a disjoint pair of $\mathfrak{J}$-closed subsets of X. Since the space $(X, \mathfrak{J})$ is normal there exists $\mathfrak{J}$-open sets G and H such that

$$f^{-1}(L) \subset G,\ f^{-1}(M) \subset H \text{ and } G \cap H = \phi$$

But
$$f^{-1}(L) \subset G \Rightarrow f(f^{-1}(L)) \subset f(G)$$
$$\Rightarrow L \subset f(G)$$

Similarly, $M \subset f(H)$. Also since f is an open map, $f(G)$ and $f(H)$ are $\mathcal{V}$-open subsets of Y such that

$$f(G) \cap f(H) = f(G \cap H) = f(\phi) = \phi.$$

Thus there exist $\mathcal{V}$-open subsets $G_1 = f(G)$ and $H_1 = f(H)$ of Y such that $L \subset G_1$, $M \subset H_1$ and $G_1 \cap H_1 = \phi$.

It follows that $(Y, \mathcal{V})$ is also a normal space, Accordingly normality is a topological property.

7. a) Let $(X, \mathcal{J})$ be a regular space and $(Y, \mathcal{V})$ be a homeomorphic image of $(X, \mathcal{J})$ under a homeomorphism f. To show that $(Y, \mathcal{V})$ is also a regular space. Let F be a $\mathcal{V}$-closed subset of Y and let q be a point of Y such that $q \notin F$. Since f is a one-one onto map there exists $p \in X$ such that

$$f(p) = q \iff f^{-1}(q) = p$$

Again since f is $\mathcal{J}$-$\mathcal{V}$ continuous. $f^{-1}(F)$ is $\mathcal{J}$-closed.

Also $\quad q \notin F \Rightarrow f^{-1}(q) \notin f^{-1}(F)$

$$\Rightarrow p \notin f^{-1}(F)$$

Thus $f^{-1}(F)$ is a $\mathcal{J}$-closed set and p is a point of X such that $p \notin f^{-1}(F)$. Since $(X, \mathcal{J})$ is a regular space there exists $\mathcal{J}$-open sets G and H such that

$$p \in G, \; f^{-1}(F) \subset H \text{ and } G \cap H = \phi$$

Now $\quad p \notin G \Rightarrow f(p) \notin f(G) \Rightarrow q \notin f(G)$

$$f^{-1}(F) \subset H \Rightarrow f(f^{-1}(F)) \subset f(H) \Rightarrow F \subset f(H)$$

and $\quad G \cap H = \phi \Rightarrow f(G \cap H) = f(\phi) \Rightarrow f(G) \cap f(H) = \phi$

Also since f is an open map, $G_1 = f(G)$ and $H_1 = f(H)$

are V-open sets. Thus there exists V-open sets G_1 and H_1 such that

$$a \in G_1, F \subset H_1, \text{ and } G_1 \cap H_1 = \phi.$$

It follows that (Y, V) is also a regular space. Accordingly regularity is a topological property.

~~8(a)~~

(b) Let $X = \{a, b, c, d, e\}$

and $A = \{\{a\}, \{a, b, c\}, \{c, d\}\}$

First compute the class B of all finite intersection of sets in A.

$$B = \{\{a\}, \phi, \{c\}, \{a, b, c\}, \{c, d\}, X\}$$

Taking the unions of members of B gives the class

$$T = \{X, \phi, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}\}$$

which is the topology on X generated by A.

8 a) Let (X, T) be a normal T_1-space. And let A be a closed subset of X and T_A be the relative topology we have to show that (A, T_A) is a normal T_1-space. Let L^*, M^* be disjoint T_A-closed subset of A. Then there exist T-closed subsets L, M of X such that $L^* = L \cap A$ and $M^* = M \cap A$. Since A is T-closed, it follows that L^*, M^* are disjoint T-closed subset of X. Then by normality of X, there exist T-open subset G, H of X. $L^* \subset G$, $M^* \subset H$ and $G \cap H = \phi$. Since $L^* \subset A$ and $M^* \subset A$, these relations imply that

$$L^* \subset G \cap A, \; M^* \subset H \cap A \text{ and } (G \cap A) \cap (H \cap A) = \phi.$$

Hence (A, T_A) is normal T_1-space.

8 b) Let $(X, \mathcal{J})$ be a Hausdorff space and let f be a one-one, open mapping of $(X,\mathcal{J})$ onto another space (Y, V). We shall show that (Y, V) is also Hausdorff. Let y_1, y_2 be two distinct points of Y. Since f is one-one onto map, there exists distinct points of x_1, x_2 of X such that $f(x_1)=y_1$ and $f(x_2)=y_2$. Since $(X,\mathcal{J})$ is a Hausdorff space, there exists $\mathcal{J}$-open sets G and H such that

$$x_1 \in G, \; x_2 \in H \text{ and } G \cap H = \phi.$$

Again since f is an open mapping, $f(G)$ and $f(H)$ are V-open set such that

$$y_1 = f(x_1) \in f(G), \quad y_2 = f(x_2) \in f(H)$$

and $$f(G) \cap f(H) = f(G \cap H) = f(\phi) = \phi$$

It follows that (Y, V) is also Hausdorff. Since the property of being a Hausdorff is preserved under one-one, onto, open maps, it is surely preserved under homeomorphism. Hence it is topological property

— x —

(KNVV VARA PRASAD)